APPLICATION OF CONSERVATION LAWS TO DERIVATION OF CONDITIONS OF STABILITY FOR STATIONARY FLOWS OF AN IDEAL FLUID

V. A. Vladimirov

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In this article we study the properties of the integrals of motion of an ideal incompressible fluid useful from the point of view of stability problems. From the integrals of motion we construct a functional whose stationary point is the given steady-state flow of the fluid. In cases where the signature of the second variation is defined this functional can take the role of the Lyapunov function [1, 2] in the problem of determining the classes of stable flows. The analysis of the second variation leads to the conclusion that in general its signature is undefined. It is however defined for special classes of motion characterized by the presence of one of the symmetry types. The general expressions for the second variation are integrals of the equations of motion linearized for the given stationary flow. Various forms of these integrals are analyzed and corresponding integrals are given for the flow of a fluid continuously stratified in density in the gravitational field. We discuss the hydrodynamical meaning of the results and their connection with the variational principle [3, 4]. The work has a methodological character.

1. Fundamental Equations. Let us consider the three-dimensional motion of a homogeneous in density, ideal and incompressible fluid in region τ with a fixed rigid boundary $\partial\tau$. The following symbols are used: x_1, x_2, x_3 and t are the Cartesian coordinates and time; $\mathbf{n} = (n_1, n_2, n_3)$ is the normal to $\partial\tau$; p and $\mathbf{u} = (u_1, u_2, u_3)$ is the pressure and velocity field. The equations of motion are of the form

$$D\mathbf{u} = -\nabla p, \quad \operatorname{div}\mathbf{u} = 0, \quad D \equiv \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla. \tag{1.1}$$

On the boundary $\partial\tau$ the following condition assures no leaking:

$$\mathbf{u}\cdot\mathbf{n} \equiv u_i n_i = 0. \tag{1.2}$$

Repeated indices of vectors and tensors are everywhere understood as summation indices.

Let $a(\mathbf{x}, t)$ be a scalar function whose values are conserved in each fluid particle:

$$Da = 0. \tag{1.3}$$

By means of the vorticity field $\boldsymbol{\omega}$ one more function is introduced

$$\lambda(\mathbf{x}, t) \equiv (\boldsymbol{\omega}\cdot\nabla)a, \tag{1.4}$$

which also conserves its values in each fluid particle

$$D\lambda = 0. \tag{1.5}$$

In particular, the field a can be considered as one of the Lagrangian coordinates of fluid particles. The relations (1.4), (1.5) then assume the sense of one of the components of the Cauchy integral of the Euler equations [5].

The stationary solutions of the equations (1.1)-(1.5) we denote in the following manner

$$\begin{gathered}\mathbf{u} = \mathbf{U}(\mathbf{x}),\ \boldsymbol{\omega} = \boldsymbol{\Omega}(\mathbf{x}),\ p = P(\mathbf{x}),\ a = A(\mathbf{x}),\\ \lambda = \Lambda(\mathbf{x}),\ \mathbf{U} = (U_1,\ U_2,\ U_3),\ \boldsymbol{\Omega} = (\Omega_1,\ \Omega_2,\ \Omega_3).\end{gathered} \tag{1.6}$$

The functions in (1.6) satisfy, as follows from (1.1)-(1.5), the following equations and boundary conditions:

$$(\mathbf{U}\cdot\nabla)\mathbf{U} = -\nabla P,\ \operatorname{div}\mathbf{U} = 0; \tag{1.7}$$

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$$(\mathbf{U}\cdot\nabla)A = 0,\ (\mathbf{U}\cdot\nabla)\Lambda = 0; \tag{1.8}$$

$$\mathbf{U}\cdot\mathbf{n} = 0 \text{ on } \partial\tau. \tag{1.9}$$

In the general case the initial- and boundary-value problem for (1.1)-(1.5) has two integrals: the first one is energy

$$2E \equiv \int_\tau u_i u_i d\tau = \text{const},$$

and the second is defined using an arbitrary function of two variables $\Phi(a, \lambda)$:

$$\int_\tau \Phi(a, \lambda)\, d\tau = \text{const}.$$

From these integrals the following invariant functional is constructed:

$$F \equiv \int_\tau \left[\frac{1}{2} u_i u_i + \Phi(a, \lambda)\right] d\tau. \tag{1.10}$$

Below we show that with an appropriate choice of the function Φ the steady-state flow (1.6)-(1.9) is the stationary point of the functional F. On the basis of the expression for the second variation the properties of F in vicinity of this point can be studied. There are applications to the stability theory.

2. Stationary Flows. Let

$$A(\mathbf{x}) = \text{const}, \qquad \Lambda(\mathbf{x}) = \text{const} \tag{2.1}$$

be two families of stream surfaces (1.8). By definition, the stream surface is such that at each of its points the velocity is in the tangent plane. It is assumed that in the whole region τ $\nabla A \times \nabla\Lambda \neq 0$ and every stream line of the flow (1.6) is an intersection of a pair of surfaces (2.1). The Bernoulli integral of Eqs. (1.7) is written in the form

$$\frac{1}{2} U^2 + P = H(A, \Lambda), \tag{2.2}$$

where $U \equiv |\mathbf{U}|$; H is the Bernoulli constant, invariant along each stream line. It follows from (2.2), (1.7)

$$\mathbf{U}\times\mathbf{\Omega} = \nabla H = H_A \nabla A + H_\Lambda \nabla\Lambda. \tag{2.3}$$

The indices A and Λ denote the corresponding partial derivatives. By performing the vector product of (2.3) successively with ∇A and $\nabla\Lambda$ one obtains

$$\Lambda\mathbf{U} = H_\Lambda \nabla A \times \nabla\Lambda,\ [(\mathbf{\Omega}\cdot\nabla)\Lambda]\mathbf{U} = -H_A \nabla A \times \nabla\Lambda. \tag{2.4}$$

From (2.4) follows

$$(\mathbf{\Omega}\cdot\nabla)\Lambda = -H_A \Lambda / H_\Lambda. \tag{2.5}$$

By introducing a function $\Psi(A, \Lambda)$ such that

$$\Psi_\Lambda \doteq H_\Lambda / \Lambda, \tag{2.6}$$

one can write any of the equalities (2.4) in the form

$$\mathbf{U} = \nabla A \times \nabla\Psi. \tag{2.7}$$

The pair of functions $A(\mathbf{x})$ and $\Psi(\mathbf{x})$ constitutes the generalized stream function of a given three-dimensional flow (see [6], p. 62).

3. Conditions of Extremum. The first variation of the functional (1.10), taken for the solution (1.6), is written as follows

$$\delta F = \int_\tau \left[U_i \delta u_i + \Phi_A \delta a + \Phi_\Lambda \left(\delta\omega_i \frac{\partial A}{\partial x_i} + \Omega_i \frac{\partial \delta a}{\partial x_i}\right)\right] d\tau. \tag{3.1}$$

By means of (2.7) the integral (3.1) takes on the form

$$\delta F = \int_{\partial\tau} \Psi \left(e_{ihm} \frac{\partial A}{\partial x_h} \delta u_i + \Omega_m \delta a\right) n_m dS + \int_\tau \left[(\Phi_\Lambda - \Psi) \frac{\partial A}{\partial x_i} \delta\omega_i + (\Phi_A - \mathbf{\Omega}\cdot\nabla\Phi_\Lambda)\, \delta a\right] d\tau. \tag{3.2}$$

Both expressions in parentheses in the volume integral are zero if the arbitrary function Φ is selected so that it satisfies the equation

$$\Phi + H - \Phi_\Lambda \Lambda = 0. \tag{3.3}$$

Indeed, taking the derivative of (3.3) with respect to the variable Λ and using (2.6) we get

$$\Phi_{\Lambda\Lambda} = H_\Lambda/\Lambda = \Psi_\Lambda. \tag{3.4}$$

Integration of (3.4) leads to the equality

$$\Phi_\Lambda = \Psi, \tag{3.5}$$

where the arbitrary function of A arising in integration is included in Ψ. Such an inclusion is possible by virtue of the definition of Ψ (2.6). Taking (2.5) one can also show that

$$\Phi_A - (\mathbf{\Omega}\cdot\nabla)\Phi_\Lambda = (\Phi + H - \Lambda\Phi_\Lambda)_A = 0. \tag{3.6}$$

If Eqs. (3.5), (3.6) are satisfied the volume integral in (3.2) vanishes.

Further, we take such boundary conditions and limitations on the variations that are sufficient for the surface integral in (3.2) to vanish. We give three types of such conditions where we utilize the remaining arbitrary choice of the functions $\Phi(\Lambda, A)$, $A(\mathbf{x})$, the properties of the flow (1.6) and its boundary $\partial\tau$.

1. If the boundary $\partial\tau$ is composed of one closed surface, then the function $\Phi(\Lambda, A)$ is chosen so that

$$\Psi = \Phi_\Lambda = 0 \text{ on } \partial\tau. \tag{3.7}$$

The possibility of such choice depends on two factors. First, from the adopted definitions it follows that on $\partial\tau$ Λ and A are functionally dependent: $\varphi(\Lambda, A) = 0$. Second, the function $\Phi(\Lambda, A)$, being the solution of Eq. (3.3), is determined to within the term $\Lambda f(A)$, where f(A) is an arbitrary function of the argument A.

2. The function A(x) is chosen so that the boundary $\partial\tau$ is one of the surfaces A = const. Then in order for (3.2) to vanish it is appropriate to adopt also $\delta a = 0$ on $\partial\tau$:

$$A = \text{const}, \quad \delta a = 0 \text{ on } \partial\tau. \tag{3.8}$$

Consideration of a broader class of variations δa is unnecessary from the point of view of the stability theory since during actual motion the equality a = const on $\partial\tau$ is always satisfied provided it was satisfied in the initial moment.

3. Let the intersection lines of the surfaces A = const and $\partial\tau$ be closed curves on which the velocity circulation is denoted by $\Gamma(A)$. Then it suffices to satisfy the equalities

$$\mathbf{\Omega}\cdot\mathbf{n} = 0, \ \mathbf{n}\times\nabla A = \mathrm{f}(A), \ \delta\Gamma(A) = 0 \text{ on } \partial\tau. \tag{3.9}$$

Here the function f(A) can be chosen arbitrarily. Conditions of the type (3.9) are conveniently used in formulation of the variational principle for two-dimensional (planar, axisymmetric, etc.) flows where the requirement $\delta\Gamma = 0$ reduces to constancy in varying the velocity circulation on the contour of the boundary [7-9].

In this manner if for Φ in (1.10) one takes a function satisfying Eq. (3.3) and adopts one of the groups of the boundary conditions 1-3 (or their combination), then the functional F has for the given flow (1.6) its stationary value.

4. The Second Variation. For δ^2F we have

$$2\delta^2F = \int [\delta u_i\delta u_i + \Phi_{AA}(\delta a)^2 + \Phi_{\Lambda\Lambda}(\delta\lambda)^2 + 2\Phi_{A\Lambda}\delta a\delta\lambda + 2\Phi_\Lambda\delta^2\lambda]\, d\tau, \tag{4.1}$$

where $\delta\lambda \equiv \Omega_i \frac{\partial\delta a}{\partial x_i} + \delta\omega_i\frac{\partial A}{\partial x_i}$; $\delta^2\lambda \equiv \delta\omega_i\frac{\partial\delta a}{\partial x_i}$.

By simple operations and by using (3.3) we obtain

$$2\delta^2F = \int_\tau \Big[\delta u_i\delta u_i - \frac{2}{\Lambda}\mathbf{U}\times\mathbf{\Omega}\delta\omega\delta a + (H_\Lambda\delta\lambda + H_A\delta a)^2/\Lambda H_\Lambda\Big]\, d\tau + \int_{\partial\tau}\Big[\frac{\Phi_A}{\Lambda}\Omega_i\delta a + 2\Phi_\Lambda\delta\omega_i\Big]\delta a n_i dS. \tag{4.2}$$

The derivatives Φ_A and Φ_Λ in (4.2) are expressed by the function Ψ with the aid of (3.5) and (3.6):

$$\Phi_A = (\mathbf{\Omega}\cdot\nabla)\Psi. \tag{4.3}$$

The integrands (4.1) and (4.2) are quadratic forms of the variations δu and δa. In the cases of a definite signature of these forms the flow (1.6) is stable. Unfortunately, the term proportional to $\delta\omega\,\delta a$ in the volume integral in the general case rules out such a possibility. A definite signature takes place only in particular cases.

5. Planar Flows. Let the principal flow (1.6) and its perturbations (variations) be planar so that all functions entering (4.2) are independent of the Cartesian coordinate x_3. Then one chooses

$$A \equiv x_3,\ \Psi = \Psi(x_1,\ x_2), \tag{5.1}$$

whereby Ψ assumes the meaning of the stream function of planar motion:

$$U_1 = -\partial\Psi/\partial x_2,\ U_2 = \partial\Psi/\partial x_1,\ U_3 \equiv 0.$$

From (5.1) follows that $\Lambda = \Omega_3 \equiv \Omega$, and from (4.3), (3.3)

$$\Phi = \Phi(\Omega),\ H = H(\Omega),\ \Psi = \Psi(\Omega). \tag{5.2}$$

As in planar motion the trajectories of the fluid particles do not leave the plane x_3 = const, where, according to (5.1), also A = const, one can assume $\delta a \equiv 0$. As a result (4.2) reduces to the form

$$2\delta^2 F = \int\left[(\delta u_1)^2 + (\delta u_2)^2 + \frac{\nabla\Psi}{\nabla\Omega}(\delta\omega_3)^2\right]dx_1 dx_2. \tag{5.3}$$

The symbol $\nabla\Psi/\nabla\Omega$ is interpreted according to (5.2). In the considered case of planar motion the original functional F (1.10) as well as the expression for its second variation (5.3) agree with the formulas obtained earlier [7]. The sufficient conditions of nonlinear stability, related to the definite signature of the form (5.3), are given in [8].

6. Other Kinds of Motion with Symmetries. An expression for (4.2) analogous to (5.3) can be obtained for motions with axial or helical symmetry. To consider the axially symmetric motion we introduce cylindrical coordinates r, θ, z

$$x_1 = z,\ x_2 = r\cos\theta,\ x_3 = r\sin\theta$$

and choose

$$A = \theta,\ \Psi = \Psi(r,\ z). \tag{6.1}$$

Here Ψ is the stream function of the axisymmetric motion; $rU_z = -\partial\Psi/\partial r$, $rU_r = \partial\Psi/\partial z$, $U_\theta \equiv 0$; indices r, θ, z denote the corresponding velocity components. From (6.1) it follows that $\Lambda = (\boldsymbol{\Omega}\cdot\nabla)A = \Omega/r$, $\Omega \equiv \Omega_\theta$ and a condition of the type of (5.2) gives $\Psi = \Psi(\Omega/r)$. Since for axisymmetric motion the fluid particles are restricted to the planes θ = const, then analogously to the foregoing one can take $\delta a \equiv 0$. The second variation (4.2) takes the form

$$2\delta^2 F = \int\left[(\delta u_r)^2 + (\delta u_z)^2 + \frac{\nabla\Psi}{\nabla(\Omega/r)}\left(\frac{\delta\omega_\theta}{r}\right)^2\right]dr dz. \tag{6.2}$$

A somewhat more complicated expression obtains for the motions with helical symmetry where one should assume $A = a\theta + bz$ with constant a and b. The conditions of nonlinear stability due to the definite signature of the form (6.2) and the analogous expression for the case of helical symmetry are given in [9].

7. Second Variation as Integral of the Linear Problem. The remarkable property of the quantity (4.1), (4.2) is its invariance with respect to the linearized equations of motion. In order to formulate and prove this fact it is convenient to introduce the following symbols. Let $\mathbf{u}, \boldsymbol{\omega}, a, \lambda$ be infinitesimally small in amplitude perturbations of the flow (1.6), described by the equations

$$\begin{gathered}
Du_i + \frac{\partial U_i}{\partial x_\alpha}u_\alpha = -\frac{\partial p}{\partial x_i},\\
Da + \frac{\partial A}{\partial x_\alpha}u_\alpha = 0,\quad D\lambda + \frac{\partial\Lambda}{\partial x_\alpha}u_\alpha = 0,\\
D\omega_i + \frac{\partial\Omega_i}{\partial x_\alpha}u_\alpha = \Omega_\alpha\frac{\partial u_i}{\partial x_\alpha} + \omega_\alpha\frac{\partial U_i}{\partial x_\alpha},\\
D \equiv \frac{\partial}{\partial t} + U_\alpha\frac{\partial}{\partial x_\alpha},
\end{gathered} \tag{7.1}$$

where $\Lambda \equiv \Omega_i\frac{\partial A}{\partial x_i}$, $\lambda \equiv \Omega_i\frac{\partial a}{\partial x_i} + \omega_i\frac{\partial A}{\partial x_i}$, and the fields $\mathbf{U}$ and $\mathbf{u}$ satisfy the no-leak conditions on $\partial\tau$

$$U_i n_i = u_i n_i = 0. \tag{7.2}$$

Now we write the expression

$$2F \equiv \int_\tau \left(u_i u_i + \Phi_{AA} a^2 + \Phi_{\Lambda\Lambda} \lambda^2 + 2\Phi_{A\Lambda} a\lambda + 2\Phi_\Lambda \omega_i \frac{\partial a}{\partial x_i} \right) d\tau, \tag{7.3}$$

obtainable from (4.1) by removing the symbols δ. The function $\Phi(A, \Lambda)$ in (7.3) satisfies (3.3) as before. By direct calculation one shows that the derivative dF/dt, calculated by means of (7.1), (7.2), reduces to an integral of a divergence form which is then transformed into a surface integral yielding

$$2\frac{dF}{dt} = \int_{\partial\tau} \left\{ \Psi \left[\left(\Omega_\alpha \frac{\partial u_i}{\partial x_\alpha} - u_\alpha \frac{\partial \Omega_i}{\partial x_\alpha} \right) a - \omega_i u_\alpha \frac{\partial A}{\partial x_\alpha} \right] - a\Omega_i u_\alpha \frac{\partial \Psi}{\partial x_\alpha} \right\} n_i dS.$$

If either (3.7) or (3.8) should hold on $\partial\tau$, then this surface integral vanishes and F is conserved (7.3). For the motions with symmetry the integrals of the linearized equations of motion are obtained from (5.3), (6.2).

8. Integral of Motion in Terms of Lagrangian Displacements. The existence of the integral of the type (7.3) for (7.1) can be expected on the basis of time independence of the coefficients (7.1). The integral (7.3) plays the role of energy. In this treatment the dependence of (7.2) on the fields of scalar admixtures a and λ will be unexpected and artificial. It turns out, however, that the role of these fields consists in fact in an implicit introduction of the Lagrangian variables.

The Lagrangian displacements of the fluid particles are most naturally introduced by considering the connections between the Lagrangian **X** and Eulerian **x** coordinates of the principal and perturbed motion. Let the function **x**(**X**, t) and the inverse **X**(**x**, t) describe the principal (unperturbed) motion of the fluid. Further, let

$$\mathbf{x}_1(\mathbf{X}, t) = \mathbf{x}(\mathbf{X}, t) + \boldsymbol{\xi}(\mathbf{X}, t) \tag{8.1}$$

be the perturbed motion with the displacement field $\boldsymbol{\xi}(\mathbf{X}, t)$. The latter, with the aid of the function **X**(**x**, t), is expressed in the form $\boldsymbol{\xi}(\mathbf{x}, t)$. By the same token the field of the Lagrangian displacements of fluid particles is represented as a function of the Eulerian coordinates. Let now Q(**x**, t) be an arbitrary characteristic of the medium defined on the perturbed motion (8.1), and Q_0(**x**, t) be the same characteristic on the unperturbed motion. The difference $q \equiv Q(\mathbf{x}, t) - Q_0(\mathbf{x}, t)$ is called the Eulerian perturbation of the field Q. At the same time the difference $\Delta Q \equiv Q(\mathbf{x}_1, t) - Q_0(\mathbf{x}, t)$ with $\mathbf{x}_1 = \mathbf{x} + \boldsymbol{\xi}(\mathbf{x}, t)$ (8.1) is called the Lagrangian perturbation of the characteristic Q. If the perturbations are small the following relation is valid to first order

$$\Delta Q = q + \xi_\alpha \frac{\partial Q}{\partial x_\alpha}. \tag{8.2}$$

If the quantity Q is conserved in each fluid particle and the perturbations consist solely of displacements of particles, then $\Delta Q = 0$ and

$$q = -\xi_\alpha \frac{\partial Q}{\partial x_\alpha}. \tag{8.3}$$

For the velocity field we have the relations

$$\Delta u_i = u_i + \xi_\alpha \frac{\partial U_i}{\partial x_\alpha}, \quad \Delta u_i = \frac{d\xi_i}{dt} = \frac{\partial \xi_i}{\partial t} + U_\alpha \frac{\partial \xi_i}{\partial x_\alpha}, \tag{8.4}$$

where Δu_i is the Lagrangian perturbation of velocity and U_i is the velocity field of the unperturbed motion. The first of equalities (8.4) follows from (8.2), the second one from the definition of the field $\boldsymbol{\xi}$. From (8.4) it follows that

$$\frac{\partial \xi_i}{\partial t} = u_i + \{\mathbf{U}, \boldsymbol{\xi}\}_i \equiv u_i + \xi_\alpha \frac{\partial U_i}{\partial x_\alpha} - U_\alpha \frac{\partial \xi_i}{\partial x_\alpha}. \tag{8.5}$$

The braces $\{\mathbf{A}, \mathbf{B}\}$ denote the Poisson brackets of the vector fields **A** and **B**. The conditions of no leak at the boundary remove the normal displacements on $\partial\tau$, which yields to first order

$$\boldsymbol{\xi} \cdot \mathbf{n} = 0. \tag{8.6}$$

The equations for the perturbed vorticity (7.1) are rewritten in the form

$$\omega_t = \{\mathbf{U}, \omega\} + \{\mathbf{u}, \Omega\}. \tag{8.7}$$

Substituting in (8.7) the expression for $\mathbf{u}$ from (8.5), after some transformations (using the Jacobi identities for the Poisson brackets and equality $\{\mathbf{U}, \Omega\} = 0$ for $\alpha \equiv \omega - \{\xi, \Omega\}$ one obtains the equation $\alpha_t = \{\mathbf{U}, \alpha\}$. If one chooses $\alpha \equiv 0$, then the following relation will hold

$$\omega = \{\xi, \Omega\}. \tag{8.8}$$

This choice is equivalent to defining the initial value $\xi(\mathbf{x}, 0)$ in a given field $\omega(\mathbf{x}, 0)$.

For the perturbations of the fields a and λ (7.1) we obtain in correspondence with (8.3)

$$a = -\xi_i \frac{\partial A}{\partial x_i}, \quad \lambda = -\xi_i \frac{\partial \Lambda}{\partial x_i}. \tag{8.9}$$

One can verify that the relations (8.9), (8.8), (8.5) are consistent with Eqs. (7.1) and with each other. In particular, if (8.5) is satisfied it guarantees the validity of the equations for a, λ, and ω from (7.1), and (8.8) leads to the situation where the second of Eqs. (8.9) follows from the first. A discussion of the Lagrangian displacements in a similar framework is given in [10, Sec. 13].

Substitution of (8.8), (8.9) into (7.3) and simple transformations lead to the result

$$2F = \int_\tau (u_i u_i + \omega_i e_{ikm} U_k \xi_m)\, d\tau + \int_{\partial\tau} \left(2\omega_i \Psi - \Omega_i \xi_h \frac{\partial \Psi}{\partial x_h}\right) a n_i dS. \tag{8.10}$$

If the conditions (3.7) or (3.8) are assumed to hold on $\partial\tau$, then the surface integral vanishes and (8.10) agrees with the expression for the second variation of energy given in [3].

9. Integrals of Motion for Flows of Stratified Fluids. It is shown in [11, 12] that the exact equations and boundary conditions describing translationally invariant motion of a homogeneous ideal incompressible fluid in a rotating coordinate system can be reduced by simple transformations to equations and boundary conditions for planar motion of a stratified fluid (written in the Boussinesq approximation). Using this equivalence one can formulate some of the results of this work for the stratified fluid.

In particular, an interesting form is assumed by the conserved functional, obtained from the integral of type (7.3), (8.10), for linear perturbations of the plane-parallel flow of the stratified fluid. In order to write it down we introduce a system of Cartesian coordinates x, y. Let the planar motion of the ideal incompressible stratified fluid take place in the region $0 < y < H$ in an external homogeneous gravitational field $\mathbf{g} = (0, g)$. The plane-parallel motion is given by two arbitrary functions $U(y)$ and $\rho_0(y)$:

$$U = U(y), \quad V \equiv 0, \quad \rho_0 = \rho_0(y). \tag{9.1}$$

Here U and V are the x and y components of velocity of the principal flow; $\rho_0(y)$ is its density distribution. By u, v, ρ, p, $\omega \equiv v_x - u_y$ we denote the perturbations of the x and y components of velocity, density, pressure and vortex satisfying the system of equations linearized on (9.1) and the no-leak boundary conditions:

$$\begin{gathered} Du + U'v = -p_x, \quad Dv = -p_y + \rho g, \\ D\rho + \rho_0' v = 0, \quad u_x + v_y = 0, \quad D \equiv \frac{\partial}{\partial t} + U\frac{\partial}{\partial x}; \end{gathered} \tag{9.2}$$

$$v = 0 \quad \text{for} \quad y = 0; H \tag{9.3}$$

The prime denotes ordinary derivatives with respect to y. By direct calculations one can verify that in the problem (9.2), (9.3) the following functional does not depend on time

$$\int \left(u^2 + v^2 + \frac{g\rho_0' - UU''}{(\rho_0')^2}\rho^2 - \frac{2U}{\rho_0'}\rho\omega\right) dx\, dy. \tag{9.4}$$

Moreover, one assumes that the functions u, v, ρ, ω are either periodic in x (then the integral is taken over the period), or decay fast enough for $x \to \pm\infty$. From the form of (9.4) and the Galilean invariance it follows that the following integral is also conserved

$$\int \left[\frac{U''}{(\rho_0')^2}\rho^2 + \frac{2}{\rho_0'}\rho\omega\right] dx\, dy. \tag{9.5}$$

It can be obtained independently from the functional of the type (1.10) for the momentum.

By analogy with (9.4), (9.5) one can write down the integrals of the full equations of motion of the stratified fluid linearized on (9.1) (without the Boussinesq approximation). By virtue of the equations

$$\rho_0(Du + U'v) = -p_x,\ \rho_0 Dv = -p_y + \rho g,$$
$$u_x + v_y = 0,\ D\rho + \rho_0' v = 0$$

and the boundary conditions (9.3) the following functional is conserved

$$\int \left[\rho_0 (u^2 + v^2) + \frac{(UU' + g)\rho_0' - U(\rho_0 U)''}{(\rho_0')^2} \rho^2 - \frac{2U}{\rho_0'} \sigma\rho \right] dx\, dy, \tag{9.6}$$

where $\sigma \equiv \rho_0\omega - \rho_0' u - (\rho U)_y$. From (9.6) follows an integral of the type (9.5)

$$\int \left[\frac{(\rho_0 U)''}{(\rho_0')^2} \rho^2 + \frac{2}{\rho_0'} \sigma\rho \right] dx\, dy.$$

It is apparent that for $\rho_0' g > 0$ (stable stratification in the sense of upward decreasing density) the forms (9.4), (9.6) are positive definite only for the state of rest $U \equiv 0$ (or a uniform flow $U \equiv$ const). Moreover, the second variation (9.6) takes the form [9]

$$\int \left[\rho_0 (u^2 + v^2) + \frac{g}{\rho_0'} \rho^2 \right] dx\, dy.$$

Imposing an arbitrarily small velocity shift $U' \neq 0$ destroys the positive definiteness of the forms in the integrands of (9.4), (9.6). The Richardson number $\mathrm{Ri} \equiv \rho_0' g/(U')^2$ is no criterion of the positive definiteness (and hence stability). This means that the stability criterion $\mathrm{Ri} > 1/4$ [13] is not energetic (in contrast to the Rayleigh criterion of the inflection point, resulting from (5.3), or to the "centrifugal" stability condition [4, 9]). This fact seems quite surprising if one remembers that the physical sense of Ri is usually explained by energetic representations [14].

In conclusion we make a few remarks completing the presented results and clarifying their connection with other works.

1. Beside (5.3), (6.2) there are other cases of definite signature of the second variation of the functional (1.10). These cases, like those discussed in Sections 5, 6, correspond to the motion with symmetries and satisfy the conditions of "centrifugal" nonlinear stability, obtained in [9]. For instance, for the rotationally symmetric variations of the flow with circular stream lines (with a field of the angular velocity component U(r)) one assumes in (1.10) $\Phi = \Phi(a)$, $a \equiv r u_\theta$. As a result

$$\delta^2 F = \int \left[(\delta u_r)^2 + (\delta u_z)^2 + \frac{4U^2 r}{(r^2 U^2)_r} (\delta u_\theta)^2 \right] dr\, dz,$$

where the symbols of Section 6 are used. The positive definiteness of this expression is obtained if the Rayleigh criterion of "centrifugal" stability $(U^2 r^2)_r > 0$ is satisfied. One can also check the conservation of $\delta^2 F$ by virtue of the linearized equations of the rotationally symmetric motion.

2. In [3] a variational principle is proposed according to which the energy of stationary flows is extremal among the so-called "equivortical" flows. The form of the integral (1.10) is natural from the point of view of this principle as the absolute extremum of the functional (1.10) can be considered as a conditional extremum of energy with the constraint

$$\int \Phi(a, \lambda)\, d\tau = \text{const}.$$

This constraint is akin to the condition of "equivorticity" [3], although weaker.

3. In [3] an expression is obtained for the second variation of energy for increases of the velocity field constrained by the conditions of "equivorticity"

$$\delta^2 E = \int [\delta u_i \delta u_i + \delta\omega_i e_{ikm} U_k f_m]\, d\tau$$

where f_m are the "mapping coordinates" [3]. After the removal of the symbols δ and replacement of f_m by ξ_m it turns out that this functional is conserved by virtue of the linearized equations of motion (7.1), (7.2), supplemented by the relations (8.5), (8.6), (8.8). The equations for a and λ from (7.1) are not given. This fact is noted in [3]. The aforementioned reduction of $\delta^2 F$ (4.1) to a similar expression (8.10) clarifies the fact of conservation of $\delta^2 E$ for perturbations with arbitrary initial data (and not only "equivortical").

4. A variational principle for flows of a rotating fluid inhomogeneous in density is given also in [15]. The functional of the type (1.10) is built from the energy and the integral of an arbitrary function of the argument $\boldsymbol{\omega} \cdot \nabla\rho$, where ρ is density, $\boldsymbol{\omega}$ is vorticity. As a result of the discussion in [15] one obtains a number of interesting cases. They cover, e.g., the homogeneous fluid and the planar motion of a stratified fluid, corresponding to $\boldsymbol{\omega} \cdot \nabla\rho \equiv 0$.

5. In direct calculation of the time derivative of energy by applying the linearized equations of motion (7.1) there arise everywhere the Reynolds stresses

$$\frac{d}{dt}\int \frac{u_i u_i}{2}\, d\tau = \int \frac{\partial U_i}{\partial x_k} u_i u_k d\tau.$$

The existence of the integral (7.3) means that these stresses are expressed in the form of a time derivative of the corresponding functional so that the equation of total energy is at once integrated.

6. After the transition in (9.4)-(9.6) to a spectral problem for the normal waves (the fields of perturbations proportional to $\exp[i(kx - \omega t)]$) one directly obtains the well-known spectral estimates of Singh [13, 16].

7. The addition of the scalar field a increases the number of "degrees of freedom" of the functional F and, consequently, diminishes the likelihood of existence of positive-definite forms. As is shown in Section 7, the role of a consists in an implicit introduction of the Lagrangian variables. The presence of symmetry of motion permits one to remove these variables and to obtain conservation laws in terms of the velocity and density fields only. In absence of symmetry such a removal cannot be performed. Perhaps this is the imperfection of the applied method and in general case Eqs. (7.1) have integrals that do not contain Lagrangian characteristics of the flow. Undoubtedly, finding such integrals would be of interest.

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